JOURNAL OF APPROXIMATION THEORY **18**, 363–377 (1976)

Mean Convergence of Lagrange Interpolation, I

G. P. NEVAI

Department of Mathematics, University of Paris-South, 91405 Orsay, France

Communicated by T. J. Rivlin

Received July 14, 1975

Let w be a weight function defined on $(-1, 1)$ and let $p_n(w, x) = \gamma_n(w)\, x^n + \cdots$ denote the corresponding sequence of orthonormal polynomials, that is, $\gamma_n(w) > 0$ and

$$\int_{-1}^{1} p_n(w, t)\, p_m(w, t)\, w(t)\, dt = \delta_{nm}.$$

Further, let $1 > x_{1n}(w) > x_{2n}(w) > \cdots > x_{nn}(w) > -1$ be the zeros of $p_n(w, x)$. For a continuous function f on $[-1, 1]$, the Lagrange interpolation polynomial $L_n(w, f)$ is defined to be the unique algebraic polynomial of degree at most $n - 1$ coinciding with f at the nodes $x_{kn}(w)$ $(k = 1, 2,..., n)$. The theory of Lagrange interpolation has a very long history and without going into details we mention that the mere continuity of f does not guarantee uniform or even pointwise convergence of $L_n(w, f)$ as $n \to \infty$. For this reason it is more practical to consider the convergence of $L_n(w, f)$ in weighted L^p spaces, at least when we are interested in convergence of $L_n(w, f)$ for every continuous function f. In order to formulate the problem we are dealing with more precisely, let us consider the space of continuous functions on $[-1, 1]$ in two examples. One of them is C with the usual maximum norm and the other is C_v^p where the distance between two functions f and g is defined by

$$d(f, g)_{v,p} = \left[\int_{-1}^{1} |f(t) - g(t)|^p\, v(t)\, dt\right]^{1/\max(1,p)},$$

where $0 < p < \infty$ and v is a nonnegative, not almost everywhere vanishing, integrable function. For $p \geqslant 1$, of course, C_v^p is a normed space and we can write $d(f, 0)_{v,p} = \|f\|_{v,p}$. When $v \equiv 1$ we simply write C^p, $\|f\|_p$, etc.

Erdös and Turán [6] have shown that for every $f \in C$ the polynomials $L_n(w, f)$ converge to f in C_w^2 and consequently also in C_w^p with $0 < p < 2$. It follows from this result also that $L_n(w, f)$ converges to f in every C_v^p if

$0 < p < 2$ and $v^{p/(2-p)}w^{-p/(2-p)}$ is integrable. For other values of p, Erdös and Feldheim [5] and Marcinkiewicz [13] have proved that for the case of the Tschebisev weight function $w^{(-1/2,-1/2)}$, where

$$w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta \qquad (\alpha > -1, \beta > -1),$$

$L_n(w^{(-1/2,-1/2)}, f)$ converges in every $C_w{}^p$ with $p > 0$. On the other hand Feldheim [9] has proved that there exists a function $f \in C$ such that $L_n(w^{(1/2,1/2)}, f)$ does not converge to f in $C^4_{W^{(1/2,1/2)}}$. For this reason Erdös and Turán raised the question: For a given w to find all the values $p = p(w)$ for which $L_n(w, f)$ converges to f in $C_w{}^p$ for every $f \in C$. This problem was later modified by Freud, who in his book [11] suggested the investigation of convergence of $L_n(w, f)$ in some $C_v{}^p$ spaces where v does not necessarily coincide with w.[1] Let us remark that Erdös and Feldheim's and Marcinkiewicz's results obviously imply that the Lagrange interpolation taken at the Tschebisev abscissas converges in every $C_v{}^p$ if $p > 0$ and v^ϵ is integrable with some $\epsilon > 1$.

The first general result giving a partial answer to the problem of Erdös and Turán was obtained by Askey [1, 2], who, considering the case of the Jacobi weight functions $w^{(\alpha,\beta)}$ for many (but not every) values of α and β, managed to prove convergence and divergence theorems. In particular, he proved that $L_n(w^{(\alpha,\beta)}, f)$ converges to f in $C^p_{W^{(\alpha,\beta)}}$ when $\alpha, \beta \geqslant -\frac{1}{2}$ for every $f \in C$ if

$$0 < p < \min\left\{\frac{4(\alpha+1)}{2\alpha+1}, \frac{4(\beta+1)}{2\beta+1}\right\}$$

and if

$$p > \max\left\{\frac{4(\alpha+1)}{2\alpha+1}, \frac{4(\beta+1)}{2\beta+1}\right\}$$

then one can find a continuous f such that $L_n(w^{(\alpha,\beta)}, f)$ diverges in $C^p_{W^{(\alpha,\beta)}}$. In [1], Askey also formulated a conjecture concerning Freud's problem for $w = w^{(\alpha,\beta)}$ and $v = w^{(a,b)}$.

The aim of this paper is to prove Askey's conjecture in a more general settlement, and with this, to solve almost completely Erdös and Turán's problem for the case of weight functions which are similar, in the sense described later, to the Jacobi weight functions. The words "almost completely" mean that for such weight functions w we can find a number $p_0 = p_0(w)$ such that $L_n(w, f)$ converges to f in every $C_w{}^p$ if $0 < p < p_0$ and there exists a continuous $f = f_p$ such that $L_n(w, f)$ diverges in $C_w{}^p$ if $p > p_0$. We are not able to say anything about the case when $p = p_0$.

[1] The case when $v(x) \equiv 1$ was considered earlier by Turán [16] and Erdös [4].

Before the main results let us introduce some notations and mention some results which will be needed later. Every constant appearing in the estimates will be denoted by A, and they are nonnegative and take different values in different estimates. We write $a_n \sim b_n$ if for every n the ratio a_n/b_n is between two positive constants. The notations $a(x) \sim b(x)$, $a_n(x) \sim b_n(x)$ have similar meanings. We say that w is a generalized Jacobi weight and write $w \approx w^{(\alpha,\beta)}$ if $w(x) \sim w^{(\alpha,\beta)}(x)$, $w/w^{(\alpha,\beta)} \in C$, and the modulus of continuity $\omega(\delta)$ of $w/w^{(\alpha,\beta)}$ satisfies the condition

$$\int_0^1 (\omega(\delta)/\delta)\, d\delta < \infty.$$

$\mathbf{P}_n$ denotes the set of algebraic polynomials with real coefficients of degree at most n. The Christoffel functions $\lambda_n(w)$ corresponding to the weight w are defined as

$$\lambda_n(w, x) = \min_{\substack{\Pi \in \mathbf{P}_{n-1} \\ \Pi(x) \neq 0}} \Pi^{-2}(x) \int_{-1}^{1} \Pi^2(t)\, w(t)\, dt, \tag{1}$$

or—what is the same—

$$\lambda_n(w, x) = \left[\sum_{k=0}^{n-1} p_k^2(w, x)\right]^{-1}. \tag{2}$$

It is also well known that

$$\frac{1}{\lambda_n(w, x)} = \sum_{k=1}^{n} \frac{l_{kn}^2(w, x)}{\lambda_n(w, x_{kn}(w))}, \tag{3}$$

where $l_{kn}(w)$ are the fundamental polynomials of Lagrange interpolation at the zeros of $p_n(w)$, that is,

$$l_{kn}(w, x) = \frac{p_n(w, x)}{p_n'(w, x_{kn}(w))(x - x_{kn}(w))}, \tag{4}$$

or in another form,

$$l_{kn}(w, x) = \frac{\gamma_{n-1}(w)}{\gamma_n(w)}\, \lambda_n(w, x_{kn}(w))\, p_{n-1}(w, x_{kn}(w))\, \frac{p_n(w, x)}{x - x_{kn}(w)}. \tag{5}$$

By the Gauss–Jacobi mechanical quadrature formula the identity

$$\int_{-1}^{1} \Pi(t)\, w(t)\, dt = \sum_{k=1}^{n} \Pi(x_{kn})\, \lambda_n(x_{kn}) \tag{6}$$

holds for every $\Pi \in \mathbf{P}_{2n-1}$, where $x_{kn} \equiv x_{kn}(w)$ and $\lambda_n(x_{kn}) \equiv \lambda_n(w, x_{kn}(w))$. Between the zeros of $p_n(w, x)$ and the Christoffel function $\lambda_n(w, x)$ there exist strong connections which are called the Markov–Stieltjes inequalities. They are

$$\sum_{k=i+1}^{n} \lambda_n(x_{kn}) \leqslant \int_{-1}^{x_{in}} w(t)\, dt \leqslant \sum_{k=i}^{n} \lambda_n(x_{kn}). \tag{7}$$

Concerning (1)–(7) see, e.g., [11]. From the proof of (7) given in [11] it is not hard to see that we also have

$$\sum_{k=i+1}^{n} (1 \pm x_{kn})\, \lambda_n(x_{kn}) \leqslant \int_{-1}^{x_{in}} (1 \pm t)\, w(t)\, dt \leqslant \sum_{k=i}^{n} (1 \pm x_{kn})\, \lambda_n(x_{kn}). \tag{8}$$

For $f \in C$ we shall denote by $S_n(w, f)$ the nth partial sum of the Fourier series of f by the polynomials $p_k(w)$. Finally, for $1 \leqslant p < \infty$ we mean by $\| L_n(w)\|_{v,\infty,p}$ and $\| S_n(w)\|_{v,\infty,p}$ the (∞, p) norms of the linear operators $L_n(w)$ and $S_n(w)$ considered as mappings from C into $C_v{}^p$.

The main result of this paper is the following

THEOREM 1. *Let $w \approx w^{(\alpha,\beta)}$ and $v = uw^{(a,b)}$, where u^ϵ is integrable with some $\epsilon > 1$. For every $f \in C$ we have $d(L_n(w, f), f)_{v,p} \to 0$ when $n \to \infty$ if*

(i) $\max(\alpha, \beta) \leqslant -\frac{1}{2}$, $a = 0$, $b = 0$, $p > 0$.

(ii) $\min(\alpha, \beta) > -\frac{1}{2}$, *$u$ is bounded in some neighborhoods of* -1 *and* 1, $a > (2\alpha - 3)/4$, $b > (2\beta - 3)/4$, *and*

$$0 < p < \min \left\{ \frac{4(a+1)}{2\alpha + 1}, \frac{4(b+1)}{2\beta + 1} \right\}.$$

(iii) $\alpha \leqslant -\frac{1}{2} < \beta$, *$u$ is bounded in a neighborhood of* -1, $a = 0$, $b > (2\beta - 3)/4$, *and*

$$0 < p < 4(b + 1)/(2\beta + 1).$$

(iv) $\beta \leqslant -\frac{1}{2} < \alpha$, *$u$ is bounded in a neighborhood of* 1, $a > (2\alpha - 3)/4$, $b = 0$ *and*

$$0 < p < 4(a + 1)/(2\alpha + 1).$$

In order to show that the conditions imposed on p and v are close to the necessary one we shall also prove

THEOREM 2. (i) *Let $w \approx w^{(\alpha,\beta)}$ and $v = uw^{(a,b)}$. Suppose that $L_n(w,f) \to_{n\to\infty} f$ in $C_v{}^p$ for every $f \in C$. If $\alpha > -\frac{1}{2}$ and u^{-1} is bounded in a neighborhood of* 1, *then necessarily* $p \leqslant 4(a+1)/(2\alpha+1)$ *and similarly if* $\beta > -\frac{1}{2}$ *and* u^{-1} *is bounded in a neighborhood of* -1, *then* $p \leqslant 4(b+1)/(2\beta+1)$.

(ii) *For every weight* w *there exist a function* $f \in C$ *and an integrable* v *such that for every* $p > 0$ $L_n(w, f)$ *does not converge to* f *in* C_v^p.

To prove these two theorems, first we must investigate some properties of polynomials $p_n(w)$ with $w \sim w^{(\alpha,\beta)}$. Theorems 3 and 4 below may be of some interest.

LEMMA 1. *If* $w \sim w^{(\alpha,\beta)}$ *then*

$$\lambda_n(w, x) \sim \frac{1}{n}\left((1-x)^{1/2} + \frac{1}{n}\right)^{2\alpha+1}\left((1+x)^{1/2} + \frac{1}{n}\right)^{2\beta+1}$$

for $|x| \leqslant 1$.

For the case $w = w^{(\alpha,\beta)}$ this was proved in [14] and for $w \sim w^{(\alpha,\beta)}$ it follows from the well-known comparison principle (see, e.g., [11]).

LEMMA 2. *Let* $P \in \mathbf{P}_n$. *Then*

$$\max_{|x|\leqslant 1}\left| P'(x)\left((1-x)^{1/2} + \frac{1}{n}\right)^{\gamma+1}\left((1+x)^{1/2} + \frac{1}{n}\right)^{\delta+1}\right|$$

$$\leqslant An \max_{|x|\leqslant 1}\left| P(x)\left((1-x)^{1/2} + \frac{1}{n}\right)^{\gamma}\left((1+x)^{1/2} + \frac{1}{n}\right)^{\delta}\right|$$

for every real γ, δ,

$$\max_{|x|\leqslant 1-(\tau/n^2)} | P(x)\, w^{((2\alpha+1)/2p,(2\beta+1)/2p)}(x)| \leqslant An^{1/p} \| P \|_{w^{(\alpha,\beta)},p}$$

for $\tau > 0$, $\alpha > -1$, $\beta > -1$, $p \geqslant 1$ *and*

$$\| P' \|_{w^{(\alpha+(p/2),\beta+(p/2))},p} \leqslant An \| P \|_{w^{(\alpha,\beta)},p}$$

for $\alpha > -1$, $\beta > -1$, $p \geqslant 1$.

Lemma 2 was proved by Khalilova [12].

THEOREM 3. *Let* $x_{kn}(w) = \cos\theta_{kn}(w)$ *for* $k = 0, 1,..., n+1$, *where* $x_{0n}(w) = 1$ *and* $x_{n+1,n}(w) = -1$. *If* $w \sim w^{(\alpha,\beta)}$ *then* $\theta_{k+1,n}(w) - \theta_{kn}(w) \sim n^{-1}$ *for* $k = 0, 1,..., n$.

Proof. By a theorem of Erdös and Turán [8] it is enough to consider such values of k for which $|x_{kn}(w)| \geqslant \frac{1}{2}$. We shall deal only with the case

$\frac{1}{2} \leqslant x_{kn}(w) \leqslant 1$; the second one can be treated similarly. The proof will consist of five parts.

1. $\theta_{1n}(w) = O(n^{-1})$. Suppose $\theta_{1n}(w) > n^{-1}$. Then by the Markov–Stieltjes inequalities

$$\lambda_n(x_{1n}(w)) \geqslant \int_{x_{1n}(w)}^{1} w(t)\,dt,$$

that is, by Lemma 1,

$$A(1/n)\, w(x_{1n}(w))(1 - x_{1n}^2(w))^{1/2} \geqslant \int_{x_{1n}(w)}^{1} w(t)\,dt$$

and since $x_{1n}(w) \geqslant \frac{1}{2}$ obviously for big values of n we obtain

$$A(1/n)[1 - x_{1n}^2(w)]^{\alpha+1/2} \geqslant [1 - x_{1n}^2(w)]^{\alpha+1}.$$

2. $[\theta_{2n}(w)]^{-1} = O(n)$. Using the Markov–Stieltjes inequalities we get

$$\lambda_n(x_{1n}(w)) \leqslant \int_{x_{2n}(w)}^{1} w(t)\,dt.$$

Then by part 1 and Lemma 1

$$n^{-2\alpha-2} \leqslant A[1 - x_{2n}^2(w)]^{\alpha+1}.$$

3. $[\theta_{1n}(w)]^{-1} = O(n)$. We have by the Guass–Jacobi mechanical quadrature formula that for every $m \geqslant n$

$$\begin{aligned}(1 - x_{1n}(w))\, \lambda_n(x_{1n}(w)) &= \int_{-1}^{1} (1 - t)\, l_{1n}^2(w, t)\, w(t)\,dt \\ &= \sum_{k=1}^{m} (1 - x_{km}(w))\, l_{1n}^2(w, x_{km}(w))\, \lambda_m(x_{km}(w)),\end{aligned}$$

that is,

$$\begin{aligned}&(1 - x_{1n}(w))\, \lambda_n(x_{1n}(w)) \\ &\quad\geqslant (1 - x_{2m}(w)) \sum_{k=1}^{m} l_{1n}^2(w, w_{km}(w))\, \lambda_m(x_{km}(w)) \\ &\quad\quad - (1 - x_{2m}(w))\, l_{1n}^2(w, x_{1m}(w))\, \lambda_m(x_{1m}(w)) \\ &\quad= (1 - x_{2m}(w))\, \lambda_n(x_{1n}(w)) \left[1 - \frac{l_{1n}^2(w, x_{1m}(w))}{\lambda_n(x_{1n}(w))}\, \lambda_m(x_{1m}(w))\right].\end{aligned}$$

From (3) we get

$$1 - x_{1n}(w) \geqslant (1 - x_{2m}(w)) \left[1 - \frac{\lambda_m(x_{1m}(w))}{\lambda_n(x_{1m}(w))}\right].$$

Here, if we put $m = qn$, where q is big but fixed, and make use of part 1 and Lemma 1 we obtain

$$1 - x_{1n}(w) \geqslant \tfrac{1}{2}(1 - x_{2m}(w))$$

for n big enough.

4. $\theta_{k+1,n}(w) - \theta_{kn}(w) = O(n^{-1})$ *for* $0 < \theta_{kn}(w) \leqslant \pi/3$. Since $(1-t)w(t) \sim t(1-t^2)^{\alpha+1}$ for $1/10 \leqslant t \leqslant 1$, we obtain from (8)

$$\int_{x_{k+1,n}(w)}^{x_{kn}(w)} t(1-t^2)^{\alpha+1}\,dt$$
$$\leqslant A[(1 - x_{kn}^2(w))\,\lambda_n(x_{kn}(w)) + (1 - x_{k+1,n}^2(w))\,\lambda_n(x_{k+1,n}(w))]$$

and by Lemma 1 and part 1

$$[\sin\theta_{k+1,n}(w)]^{2\alpha+4} - [\sin\theta_{kn}(w)]^{2\alpha+4}$$
$$\leqslant (A/n)\{[\sin\theta_{kn}(w)]^{2\alpha+3} + [\sin\theta_{k+1,n}(w)]^{2\alpha+3}\}.$$

Consequently

$$\theta_{k+1,n}(w) - \theta_{kn}(w) \leqslant \frac{A}{n} \sup_{0 \leqslant u,v \leqslant 3\pi/4} \frac{|u-v|\,[\sin^{2\alpha+3} u + \sin^{2\alpha+3} v]}{|\sin^{2\alpha+4} u - \sin^{2\alpha+4} v|} = O\left(\frac{1}{n}\right).$$

5. $[\theta_{k+1,n}(w) - \theta_{kn}(w)]^{-1} = O(n)$ *for* $0 < \theta_{kn}(w) \leqslant \pi/3$. First we estimate $(d/dx)\, l_{kn}(w,x)$ for $x_{k+1,n}(w) \leqslant x \leqslant x_{kn}(w)$. Since by (3) $l_{kn}^2(w,x) \leqslant \lambda_n(x_{kn}(w))\,\lambda_n^{-1}(w,x)$ we obtain from Lemma 1

$$l_{kn}^2(w,x) \leqslant A\lambda_n(x_{kn}(w))n\left((1-x)^{1/2} + \frac{1}{n}\right)^{-2\alpha-1}\left((1+x)^{1/2} + \frac{1}{n}\right)^{-2\beta-1}$$

and by Lemma 2

$$\left|\frac{d}{dx}\, l_{kn}^2(w,x)\right| \leqslant An\lambda_n(x_{kn}(w))\,\lambda_n^{-1}(w,x)\left((1-x)^{1/2} + \frac{1}{n}\right)\left((1+x)^{1/2} + \frac{1}{n}\right);$$

that is, by parts 1 and 4 and Lemma 1,

$$\max_{x_{k+1,n}(w) \leqslant x \leqslant x_{kn}(w)} |(d/dx)\, l_{kn}^2(w,x)| = O(n \cdot (1 - x_{kn}^2(w))^{1/2}).$$

Finally, we observe that $1 = l_{kn}^2(w, x_{kn}(w)) - l_{kn}^2(w, x_{k+1,n}(w)) = (x_{kn}(w) - x_{k+1,n}(w))(d/dx)\, l_{kn}^2(w, x^*)$, where $x^* \in [x_{k+1,n}(w), x_{kn}(w)]$.

Theorem 3 was already known for the case $|\alpha| = |\beta| = \frac{1}{2}$. For $\alpha = \beta = -\frac{1}{2}$ it was proved by Erdös and Turán [7] and for the other combinations of α, β with $|\alpha| = |\beta| = \frac{1}{2}$, by Freud [10]. In the following we shall need one important consequence of this theorem which follows immediately from parts 1 and 4:

COROLLARY. *If* $w \sim w^{(\alpha,\beta)}$ *then* $\theta \sim \theta_{kn}(w)$ *for* $\theta_{kn}(w) \leqslant \theta \leqslant \theta_{k+1,n}(w)$ $(k = 1, 2,..., n-1)$.

THEOREM 4. *Let* $1 \leqslant p < \infty$ *and let* P *be a polynomial of degree* $m \leqslant$ const n. *If* $w \sim w^{(\alpha,\beta)}$ *then*

$$\sum_{k=1}^{n} | P(x_{kn}(w))|^p \lambda_n(x_{kn}(w)) \leqslant A \int_{-1}^{1} | P(t)|^p w(t)\, dt.$$

Proof. By Lemmas 1 and 2 and Theorem 3,

$$| P(x_{kn}(w))|^p \lambda_n(x_{kn}(w)) \leqslant A \int_{-1}^{1} | P(t)|^p w(t)\, dt$$

for $k = 1, 2,..., n$ and in particular for $k = 1$ and $k = n$. To estimate $\sum_{k=2}^{n-1} | P(x_{kn}(w))|^p \lambda_n(x_{kn}(w))$ let us observe that

$$| P(x_{kn}(w))|^p \leqslant | P(t)|^p + p \int_{x_{k+1,n}(w)}^{x_{k-1,n}(w)} | P(t)|^{p-1} | P'(t)|\, dt$$

for $x_{k+1,n}(w) \leqslant t \leqslant x_{k-1,n}(w)$, $k = 2, 3,..., n-1$. After using the Markov–Stieltjes inequalities we obtain

$$\sum_{k=2}^{n-1} | P(x_{kn}(w))|^p \lambda_n(x_{kn}(w)) \leqslant 2 \int_{-1}^{1} | P(t)|^p w(t)\, dt$$
$$+ p \sum_{k=2}^{n-1} \lambda_n(x_{kn}(w)) \int_{x_{k+1,n}(w)}^{x_{k-1,n}(w)} | P(t)|^{p-1} | P'(t)|\, dt.$$

It follows from Lemma 1 and Corollary that

$$\lambda_n(x_{kn}(w)) \sim (1/n)\, w(t)(1 - t^2)^{1/2}$$

for $x_{k+1,n}(w) \leqslant t \leqslant x_{k-1,n}(w)$, $k = 2, 3,..., n-1$. This gives us

$$\sum_{k=2}^{n-1} \lambda_n(x_{kn}(w)) \int_{x_{k+1,n}(w)}^{x_{k-1,n}(w)} | P(t)|^{p-1} | P'(t)|\, dt$$
$$\leqslant (A/n) \| P \|_{w,p}^{(p-1)} \| P' \|_{w^{(\alpha+(p/2),\beta+(p/2))},p},$$

which together with Lemma 2 shows that

$$\sum_{k=2}^{n-1} | P(x_{kn}(w))|^p \lambda_n(x_{kn}(w)) \leqslant A \int_{-1}^{1} | P(t)|^p w(t)\, dt.$$

For the case $m = n - 1$, $w = w^{(\alpha,\beta)}$, $\alpha \geqslant -\frac{1}{2}$, $\beta \geqslant -\frac{1}{2}$, and also for some other values of α and β, Theorem 4 was proved by Askey [1, 2].

THEOREM 5. *If* $w \sim w^{(\alpha,\beta)}$ *then* $\| L_n(w)\|_{v,\infty,p} \leqslant A \| S_{n-1}(w)\|_{v,\infty,p}$ *for every* $1 \leqslant p < \infty$ *and* v *integrable on* $[-1, 1]$.

Proof. By definition

$$\| L_n(w)\|_{v,\infty,p} = \sup_{\|f\|_C=1} \sup_{\|g\|_{v,q}=1} \int_{-1}^{1} L_n(w, f, t)\, g(t)\, v(t)\, dt,$$

where $q = p/(p - 1)$. Now, since $L_n(w, f) \in \mathbf{P}_{n-1}$, we get

$$\int_{-1}^{1} L_n(w, f, t)\, g(t)\, v(t)\, dt = \int_{-1}^{1} L_n(w, f, t)\, S_{n-1}(w, gv/w, t)\, w(t)\, dt$$

and by the Gauss–Jacobi mechanical quadrature formula and Theorem 4 we obtain

$$\| L_n(w)\|_{v,\infty,p} \leqslant A \sup_{\|g\|_{v,q}=1} \| S_{n-1}(w, gv/w)\|_{w,1}\,.$$

But

$$\begin{aligned} \| S_{n-1}(w, gv/w)\|_{w,1} &= \sup_{\|G\|_C=1} \int_{-1}^{1} S_{n-1}(w, gv/w, t)\, G(t)\, w(t)\, dt \\ &= \sup_{\|G\|_C=1} \int_{-1}^{1} g(t)\, S_{n-1}(w, G, t)\, v(t)\, dt \\ &\leqslant \| g \|_{v,q} \| S_{n-1}(w)\|_{v,\infty,p}\,. \end{aligned}$$

Proof of Theorem 1. Let first $u \equiv 1$ and $p \geqslant 1$. Since $L_n(w, P) \equiv P$ for every $P \in \mathbf{P}_{n-1}$ we have only to show that $\| L_n(w)\|_{v,\infty,p} \leqslant A$ under conditions (i)–(iv). But

$$\| L_n(w)\|_{v,\infty,p_1} \leqslant A \| L_n(w)\|_{v,\infty,p_2}$$

for $1 \leqslant p_1 < p_2$; consequently we must consider only large values of p. Now we make use of a result of Badkov [3] by which $\| S_n(w)\|_{w^{(a,b)},p,p}$ is uniformly bounded in n if $p > 1$ and

$$\frac{2(a + 1)}{\alpha + 1} < p < \frac{a + 1}{((\alpha + 1)/2) - \min(\frac{1}{4}, (\alpha + 1)/2)},$$
$$\frac{2(b + 1)}{\beta + 1} < p < \frac{b + 1}{((\beta + 1)/2) - \min(\frac{1}{4}, (\beta + 1)/2)},$$

where 1/0 means ∞. Since $\| S_n(w)\|_{v,\infty,p} \leqslant A \| S_n(w)\|_{v,p,p}$ we obtain from Theorem 5 that $\| L_n(w)\|_{v,\infty,p} \leqslant A$ under conditions (i)–(iv) if $p \geqslant 1$ and $u \equiv 1$. If $0 < p < 1$ we simply use the fact that

$$d(f, g)_{v,p} \leqslant Ad(f, g)_{v,1}.$$

The case when $u \not\equiv 1$ will be reduced to the case $u \equiv 1$, which has already been proved. So let u satisfy the conditions of the theorem. Then (i) is obvious. Further, in condition (ii)

$$v \leqslant A(uw^{(r+a,r+b)} + w^{(a,b)})$$

for fixed $r > 0$. Hence we can find numbers $p^* = p^*(\epsilon)$ and $q^* = q^*(p)$ so that

$$d(L_n(w, f), f)_{v,p} \leqslant A[d(L_n(w, f), f)^{q^*}_{w^{(r+a,r+b)},p^*} + d(L_n(w, f), f)_{w^{(a,b)},p}].$$

Having p^* we can choose $r = r(p^*)$ so large that

$$\lim_{n\to\infty} d(L_n(w, f), f)^{q^*}_{w^{(r+a,r+b)},p^*} = 0.$$

Thus

$$\limsup_{n\to\infty} d(L_n(w, f), f)_{v,p} \leqslant A \limsup_{n\to\infty} d(L_n(w, f), f)_{w^{(a,b)},p},$$

which proves (ii). Similarly, (iii) and (iv) follow from the fact that they are true for $u \equiv 1$.

LEMMA 3. *If $w \approx w^{(\alpha,\beta)}$ then*

$$w(x_{kn}(w))\, p^2_{n-1}(w, x_{kn}(w)) \sim (1 - x^2_{kn}(w))^{1/2}$$

for $k = 1, 2,..., n$.

Proof. The proof strongly depends on the inequality

$$|p_n(w, x)|\, ((1 - x)^{1/2} + (1/n))^{\alpha+1/2}((1 + x)^{1/2} + (1/n))^{\beta+1/2} \leqslant A \qquad (w \approx w^{(\alpha,\beta)},\ |x| \leqslant 1) \quad (9)$$

proved by Badkov [3]. Without loss of generality suppose $0 \leqslant x_{kn}(w) < 1$. Let $w_1(x) = (1 - x)\, w(x)$ and expand $(1 - x)\, p_{n-1}(w_1, x)$ into a Fourier series in the polynomials $p_k(w, x)$. It is not hard to see that

$$(1 - x)\, p_{n-1}(w_1, x) = \sum_{k=n-1}^{n} \int_{-1}^{1} (1 - t)\, p_{n-1}(w_1, t)\, p_k(w, t)\, w(t)\, dt\, p_k(w, x)$$

and consequently

$$(1 - x_{kn}(w))\, p_{n-1}(w_1, x_{kn}(w)) = (\gamma_{n-1}(w)/\gamma_{n-1}(w_1))\, p_{n-1}(w, x_{kn}(w)).$$

Since $w_1 \approx w^{(\alpha+1,\beta)}$ we obtain from Theorem 3 and (9)

$$w(x_{kn}(w))\, p_{n-1}^2(w, x_{kn}(w)) \leqslant A(1 - x_{kn}(w))^{1/2}[\gamma_{n-1}(w_1)/\gamma_{n-1}(w)]^2.$$

By a well-known theorem of Szegö (see [15, Theorem 12.7.1]) we have

$$\lim_{n\to\infty} \frac{\gamma_{n-1}(w_1)}{\gamma_{n-1}(w)} = \exp\left\{-\frac{1}{2\pi}\int_{-1}^{1} \log(1-t)\frac{dt}{(1-t^2)^{1/2}}\right\} = 2^{1/2}.$$

Thus the inequality

$$w(x_{kn}(w))\, p_{n-1}^2(w, x_{kn}(w)) \leqslant A(1 - x_{kn}^2(w))^{1/2}$$

is proved. On the other hand by (4) and (5)

$$\frac{\gamma_{n-1}(w)}{\gamma_n(w)}\, \lambda_n(x_{kn}(w))\, p_{n-1}(w, x_{kn}(w)) = \frac{1}{p_n'(w, x_{kn}(w))}$$

and using Lemma 1 and Theorem 3 we obtain

$$\begin{aligned}&[w(x_{kn}(w))\, p_{n-1}^2(w, x_{kn}(w))]^{-1}\\ &\quad\leqslant A w^{(\alpha+1,\beta+1)}(x_{kn}(w))\left[\frac{\gamma_{n-1}(w)}{\gamma_n(w)}\right]^2 \cdot n^{-2}[p_n'(w, x_{kn}(w))]^2.\end{aligned}$$

Since $\gamma_{n-1}(w) \leqslant \gamma_n(w)$ we have only to show that

$$|\, p_n'(w, x_{kn}(w))|\, w^{(\alpha/2+3/4,\beta/2+3/4)}(x_{kn}(w)) \leqslant An,$$

but this follows immediately from Lemma 2, Theorem 3, and (9).

LEMMA 4. *Let $w \approx w^{(\alpha,\beta)}$. Then*

$$|\, p_n(w, x)| \leqslant A \sum_{k=1}^{n} |\, l_{kn}(w, x)|$$

for $|x| \leqslant 1$. Furthermore, for $\alpha > -\frac{1}{2}$

$$|\, p_n(w, 1)| \sim \sum_{k=1}^{n} |\, l_{kn}(w, 1)|$$

and for $\beta > -\frac{1}{2}$

$$|p_n(w, -1)| \sim \sum_{k=1}^{n} |l_{kn}(w, -1)|.$$

Proof. By Lemma 3 $|p_{n-1}(w, x_{kn}(w))| \sim 1$ for $x_{kn}(w) \in [c, d] \subset (-1, 1)$. Thus

$$\sum_{k=1}^{n} |l_{kn}(w, x)| \geqslant A(\gamma_{n-1}(w)/\gamma_n(w)) |p_n(w, x)| \sum_{c \leqslant x_{kn}(w) \leqslant d} \lambda_n(x_{kn}(w)).$$

By the above-mentioned Szegö theorem, $\lim_{n\to\infty}(\gamma_{n-1}(w)/\gamma_n(w)) = \frac{1}{2}$ and by the Markov–Stieltjes inequalities

$$\sum_{c \leqslant x_{kn}(w) \leqslant d} \lambda_n(x_{kn}(w)) \geqslant A \cdot \int_c^d w(t)\, dt.$$

To prove the second part of the lemma we should show that

$$\sum_{k=1}^{n} \lambda_n(x_{kn}(w)) \frac{|p_{n-1}(w, x_{kn}(w))|}{1 - x_{kn}(w)} \leqslant A$$

for $\alpha > -\frac{1}{2}$ and

$$\sum_{k=1}^{n} \lambda_n(x_{kn}(w)) \frac{|p_{n-1}(w, x_{kn}(w))|}{1 + x_{kn}(w)} \leqslant A$$

for $\beta > -\frac{1}{2}$. Consider the first inequality. Clearly it is enough to prove that

$$\sum_{0 \leqslant x_{kn}(w) < 1} \lambda_n(x_{kn}(w)) \frac{|p_{n-1}(w, x_{kn}(w))|}{1 - x_{kn}(w)} \leqslant A.$$

By Lemmas 1 and 3 and Theorem 3 this is equivalent to

$$\frac{1}{n} \sum_{0 \leqslant x_{kn}(w) < 1} [1 - x_{kn}^2(w)]^{\alpha/2 - 1/4} = \frac{1}{n} \sum_{0 < \theta_{kn}(w) \leqslant \pi/2} [\sin \theta_{kn}(w)]^{\alpha - 1/2} \leqslant A.$$

But we get from Theorem 3 that $\sin \theta_{kn}(w) \sim k/n$.

Thus

$$\frac{1}{n} \sum_{0 < \theta_{kn}(w) \leqslant \pi/2} [\sin \theta_{kn}(w)]^{\alpha - 1/2} \sim n^{-\alpha - 1/2} \sum_{k=1}^{n} k^{\alpha - 1/2} \sim 1$$

if $\alpha > -\frac{1}{2}$.

LEMMA 5. *Let $\alpha > -1$, $\beta > -1$, $1 > \delta \geqslant 0$, $p > 0$ be fixed. Then for every $P \in \mathbf{P}_n$*

$$|P(1)|^p \leqslant An^{2\alpha+2} \int_\delta^1 |P(t)|^p (1-t)^\alpha \, dt$$

and

$$|P(-1)|^p \leqslant An^{2\beta+2} \int_{-1}^{-\delta} |P(t)|^p (1+t)^\beta \, dt.$$

Proof. It is enough to prove the first inequality. Since $P \equiv L_{n+1}(w^{(-3/4,-3/4)}, P)$ we obtain from (5), (9), and Theorem 3

$$|P(1)| \leqslant A \max_{|x| \leqslant 1-\tau/n^2} |P(x)| \, n^{-1/4} \sum_{k=1}^{n+1} \lambda_{n+1}(x_{k,n+1}(w^{(-3/4,-3/4)})) \times \frac{p_n(w^{(-3/4,-3/4)}, x_{k,n+1}(w^{(-3/4,-3/4)}))}{1 - x_{k,n+1}(w^{(-3/4,-3/4)})};$$

that is, by Lemmas 1 and 3 and Theorem 3

$$|P(1)| \leqslant A \max_{|x| \leqslant 1-\tau/n^2} |P(x)| \, n^{-5/4} \sum_{k=1}^{n+1} [1 - x_{k,n+1}(w^{(-3/4,-3/4)})]^{-5/8} \leqslant A \max_{|x| \leqslant 1-\tau/n^2} |P(x)|.$$

Now by Lemma 2

$$|P(1)|^p \leqslant An^{2\alpha+2} \int_{-1}^1 |P(t)|^p (1-t^2)^\alpha \, dt.$$

Substitute $t^M P((1-\delta)\, t^2 + \delta)$ here for $P(t)$, where M is a natural integer. Then

$$|P(1)|^p \leqslant A(n+M)^{2\alpha+2} \int_\delta^1 |P(t)|^p (t-\delta)^{(Mp-1)/2} \times (1-t)^\alpha \, dt \cdot (1-\delta)^{(1-Mp)/2)-\alpha}.$$

Now, let M be fixed but more than p^{-1}.

Proof of Theorem 2. (i) Consider the case when $\alpha > -\frac{1}{2}$. Let $\{\varphi_n\}$ be a sequence of linear functionals from C to $\mathbf{R}$ defined by the formula

$$\varphi_n(f) = \log n [p_n(w, 1)]^{-1} L_n(w, f, 1).$$

By Lemma 4 $\| \varphi_n \| \sim \log n$ and using the Banach–Steinhaus theorem we see that there exists a function $f_0 \in C$ such that $\varphi_n(f_0) \nrightarrow 0$ when $n \to \infty$. Thus

$$p_{n_\nu}(w, 1) \leqslant A \log n_\nu \mid L_{n_\nu}(w, f_0, 1)| \tag{10}$$

for a function $f_0 \in C$ and a suitable sequence $\{n_\nu\}$. Let $j < n$ and compare $p_j(w, 1)$ and $p_n(w, 1)$. We have by (5)

$$p_j(w, 1) = \frac{\gamma_{n-1}(w)}{\gamma_n(w)} p_n(w, 1) \sum_{k=1}^{n} \lambda_n(x_{kn}(w))$$
$$\times\, p_j(w, x_{kn}(w))\, p_{n-1}(w, x_{kn}(w))[1 - x_{kn}(w)]^{-1}.$$

Since $\gamma_{n-1}(w) \subseteq \gamma_n(w)$ we obtain from Lemmas 1 and 3, Theorem 3, and (9) that

$$p_j(w, 1) \leqslant Ap_n(w, 1) \sum_{k=1}^{n} [1 - x_{kn}^2(w)]^{-1/2}$$

and again using Theorem 3,

$$p_j(w, 1) \leqslant A \log np_n(w, 1).$$

From this, (2), and (10),

$$\lambda_{n_\nu}^{-1}(w, 1) \leqslant An_\nu(\log n_\nu)^4 \mid L_{n_\nu}(w, f_0, 1)|^2. \tag{11}$$

By the hypothesis of the theorem we have for a suitable $c < 1$

$$\int_c^1 \mid L_n(w, f_0, t)|^p(1 - t)^a \leqslant A;$$

that is, by Lemma 5,

$$\mid L_n(w, f_0, 1)| \leqslant An^{2(a+1)/p}. \tag{12}$$

It follows from Lemma 1, (11), and (12) that

$$n_\nu^{2\alpha+2} \leqslant A(\log n_\nu)^4\, n_\nu^{(1+(4(a+1)/p))}.$$

Thus

$$2\alpha + 1 \leqslant \frac{\text{og } A + 4 \log \log n_\nu}{\log n_\nu} + \frac{4(a + 1)}{p}.$$

Letting $\nu \to \infty$ we obtain $p \leqslant 4(a + 1)/(2\alpha + 1)$.

(ii) Suppose that there exists a weight w such that for every $f \in C$ and integrable v there is a $p > 0$ with $d(L_n(w, f), f)_{v,p} \to 0$ when $n \to \infty$. This means that

$$\int_{-1}^{1} | L_n(w, f, t)|^p v(t)\, dt \leqslant A$$

independently of n. Hence by the Banach–Steinhaus theorem

$$\sup_{\substack{v \in C \\ \|v\|_1 = 1}} \int_{-1}^{1} | L_n(w, f, t)|^p v(t)\, dt \leqslant A;$$

that is, $\| L_n(w, f)\|_C \leqslant A$ for every $f \in C$, which is impossible.

Acknowledgment

I would like to thank Professor Richard Askey, whose work led me to this topic, for offering several helpful suggestions.

References

1. R. Askey, Mean convergence of orthogonal series and Lagrange interpolation, *Acta Math. Acad. Sci. Hungar.* **23** (1972), 71–85.
2. R. Askey, Summability of Jacobi series, *Trans. Amer. Math. Soc.* **179** (1973), 71–84.
3. V. Badkov, Convergence in mean and almost everywhere of Fourier series in polynomials orthogonal on a segment (in Russian), *Mat. Sb.* **95 (137)** (1974), 229–262.
4. P. Erdös, Problems and results on the convergence and divergence properties of the Lagrange interpolation polynomials and some extremal problems, *Mathematica (Cluj)* **(33) 19** (1968), 65–73.
5. P. Erdös and E. Feldheim, Sur le mode de convergence pour l'interpolation de Lagrange, *Compt. Rend. Acad. Sci. Paris* **203** (1936), 913–915.
6. P. Erdös and P. Turán, On interpolation, I, *Ann. of Math.* **38** (1937), 142–155.
7. P. Erdös and P. Turán, On interpolation, II, *Ann. of Math.* **39** (1938), 703–724.
8. P. Erdös and P. Turán, On interpolation, III, *Ann. of Math.* **41** (1940), 510–553.
9. E. Feldheim, Sur le mode de convergence dans l'interpolation de Lagrange, *Soviet Math. Dokl.* **14** (1937), 327–331.
10. G. Freud, Über einen Satz von P. Erdös und P. Turán, *Acta Math. Acad. Sci. Hungar.* **4** (1953), 255–266.
11. G. Freud, "Orthogonal Polynomials," Pergamon Press, New York, 1971.
12. B. Khalilova, On some estimates for polynomials (in Russian), *Izv. Akad. Nauk. Azerb. SSR* **2** (1974), 46–55.
13. J. Marcinkiewicz, Sur l'interpolation (I), *Studia Math.* **6** (1936), 1–17.
14. G. P. Nevai, Polynomials orthonormals on the real axis with respect the weight $| x |^\alpha \exp(- | x |^\beta)$, I (in Russian), *Acta Math. Acad. Sci. Hungar.* **24** (1973), 335–342.
15. G. Szegö, "Orthogonal Polynomials," Amer. Math. Soc., Providence, R.I., 1967.
16. P. Turán, On some problems in the theory of the mechanical quadrature, *Mathematica (Cluj)* **(31) 8** (1966), 182–192.